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The integrability of the two-state problem in terms of confluent hypergeometric functions

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Abstract. The reduction of the two-state problem to the confluent hypergeometric equation via complex transformations of independent variables is considered. It is shown that all the known analytically integrable cases can be generalized to a single formula. A new class of integrable models of the two-state problem in terms of confluent hypergeometric functions is presented.

The problem of the reduction of the mathematical model of the two-state problem to some analytically integrable equation has a long history in physics dating back to the 1930s [1–8]. The exact analytic solutions found in such a manner have played an important role in the establishment of qualitative peculiarities of various phenomena, occurring in a variety of actual physical situations (the interaction of radiation with matter, atomic and nuclear collisions, etc).

Landau [1] and Zener [2] have shown that a particular form of the two-state model which avoids crossing of two energy levels can be solved in terms of confluent hypergeometric functions. Later, the reduction of the two-state problem to the confluent hypergeometric equation was used by a number of authors, mainly in the context of collision physics, magnetic resonance and atomic spectroscopy [4–8].

The very procedure of searching for exactly solvable models, applied so far, can be characterized as a 'random one' due to the absence of any systematic method to determine whether the given equation can be solved in terms of given functions. The purpose of the present paper is to propose such a systematic method consisting of a 'mapping' procedure based on the equation of invariants. We shall show that the application of the proposed method allows one to generalize all the previously known solutions and describe them by a single formula. It should also be mentioned that the approach allows one to obtain a variety of new classes of solution. We shall give an example of such a new class.

In its general form, the two-state problem is equivalent to a system of coupled first-order differential equations for probability amplitudes $c_1(t)$ and $c_2(t)$ for the two states $|1\rangle$ and $|2\rangle$

$$\mathrm{i}\hbar c_{1t} = H_{11}c_1 + H_{12}c_2$$
$$\mathrm{i}\hbar c_{2t} = H_{21}c_1 + H_{22}c_2$$

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where all the matrix elements of the Hamiltonian operator, H_{ik}, in general, depend on time, t, and the alphabetical index denotes differentiation with respect to the correspondent variable.

By the phase transformation,

$$a_{k=}c_k \exp\left(\frac{\mathrm{i}}{\hbar}\int H_{kk}\,\mathrm{d}t\right) \qquad k = 1, 2$$

this system can be written in the following canonical form:

$$\mathrm{i}\hbar a_{1t} = U \exp\left[-\mathrm{i}\int \varepsilon(t)\,\mathrm{d}t\right]a_2$$
$$\mathrm{i}\hbar a_{2t} = U \exp\left[+\mathrm{i}\int \varepsilon(t)\,\mathrm{d}t\right]a_1 \qquad (1)$$

where the functions $U = H_{21}$ and $\varepsilon = (H_{22} - H_{11})/\hbar$ are assumed to be real functions ($U > 0$).

This system is equivalent to one second-order linear equation:

$$a_{1tt} + \left(\mathrm{i}\varepsilon - \frac{U_t}{U}\right)a_{1t} + \frac{U^2}{\hbar^2}a_1 = 0. \qquad (2)$$

Let us consider the cases when this equation may be reduced to the confluent hypergeometric equation [9, 10]

$$u_{zz} + \left(\frac{B}{z} - 1\right)u_z - \frac{A}{z}u = 0 \qquad (3)$$

via a transformation of both the independent and dependent variables of the source equation (2):

$$z = z(t) \qquad (4)$$
$$a_1 = \varphi(z)\cdot u(z). \qquad (5)$$

Since the chosen form of the dependent variable transformation, equation (5), is a linear one and does not change the linearity and the uniformity of the initial equation (2), it is easily understood that the introduction of this transformation gives an additional free function, a new degree of freedom, due to which one may expect to find new integrable cases (i.e. pairs of functions U and ε for which the source equation can be reduced to the confluent hypergeometric equation) of the title problem. Moreover, as will be shown below, in the case when no such transformation is introduced, there exists no more then one family of integrable cases instead of the rich set of classes of the solutions of the equation of invariants.

Since the argument of the target equation (3) may be a complex quantity, one may consider a (one-to-one) complex-valued transformation of the real parameter t to complex z. Evidently, the complex-valuedness does not contradict any requirements although only a real transformation has been considered so far. Note that the complex-valuedness of the independent variable transformation requires the introduction of two real functions, $x(t)$ and $y(t)$ ($z = x + \mathrm{i}y$), instead of one as applied so far. Evidently, this gives us another new degree of freedom.

In the case when (4) is rewritten in the form of $t = t(z)$, the complexity of the argument of (3) means that one may consider a real function, $t = t(x, y)$, from two real arguments (where the parameters x and y are complex quantities), which transforms the source equation to a partial differential equation. It is easily understood that the chosen form (4) is more familiar so that it is convenient to apply an independent variable transformation (of form (4)) to the target equation (3) (the dependent variable transformation (5) will be applied to the source equation (2)).

Substitution of (4) into (3) yields

$$u_{tt} + \left(f z_t - \frac{z_{tt}}{z_t} \right) u_t + g z_t^2 u = 0 \tag{6}$$

where $f(z) = B/z - 1$ and $g(z) = -A/z$.

According to the theorem of invariants [10], the solutions of two second-order equations, (2) and (6), are related by expression (5) if and only if the invariants of the equations are the same,

$$\frac{U^2}{\hbar^2} - \frac{1}{2}\left(\mathrm{i}\varepsilon - \frac{U_t}{U} \right)_t - \frac{1}{4}\left(\mathrm{i}\varepsilon - \frac{U_t}{U} \right)^2 = g z_t^2 - \frac{1}{2}\left(f z_t - \frac{z_{tt}}{z_t} \right)_t - \frac{1}{4}\left(f z_t - \frac{z_{tt}}{z_t} \right)^2 \tag{7}$$

the factor φ being of the form

$$\varphi = \exp\left(-\frac{1}{2} \int \left(\mathrm{i}\varepsilon - \frac{U_t}{U} - f \right) \mathrm{d}t \right).$$

In fact, both equation (7) and the expression of φ can be obtained by application of the dependent variable transformation, equation (5), to the source equation (2), and then finding the identity of the coefficients of the obtained equation and target equation (6), and by further exclusion of function $\varphi(z)$. Note that in the case of unit φ, i.e. $a_1 = u$, the last procedure may be omitted so that one can obtain the following system of two equations:

$$\frac{U^2}{\hbar^2} = g z^2 \qquad \mathrm{i}\varepsilon - \frac{U_t}{U} = f z_t - \frac{z_{tt}}{z_t}$$

which, evidently, represents a special (and simplest) solution of the equation of invariants (7). It is easy to see that this system, used by many authors, defines, as already mentioned, a single family of (complex) pairs U and ε for which the source equation can be reduced to the confluent hypergeometric equation.

As can be easily shown (see [11]), if the functions $U^*(z)$ and $\varepsilon^*(z)$ are the solutions of equation (7) for $z \equiv t$ then the functions $U(t)$ and $\varepsilon(t)$ given by simple formulae

$$U(t) = U^*(z)\frac{\mathrm{d}z}{\mathrm{d}t} \qquad \varepsilon(t) = \varepsilon^*(z)\frac{\mathrm{d}z}{\mathrm{d}t} \tag{8}$$

which are solutions of (7) for arbitrary complex $z(t)$.

Hence, let us first set $t \equiv z$. Equation (7) becomes

$$\frac{U^{*2}}{\hbar^2} - \frac{1}{2}\left(\mathrm{i}\varepsilon^* - \frac{U_z^*}{U^*} \right)_z - \frac{1}{4}\left(\mathrm{i}\varepsilon^* - \frac{U_z^*}{U^*} \right)^2 = \frac{B}{2}\left(1 - \frac{B}{2} \right)\frac{1}{z^2} + \left(\frac{B}{2} - A \right)\frac{1}{z} - \frac{1}{4}. \tag{9}$$

It is easy to see that this equation has a solution of simple form

$$U^* = \frac{K}{z^{s/2}} \qquad s = 0, 1, 2 \tag{10}$$

$$\varepsilon^* = P + \frac{Q}{z} \tag{11}$$

where P, Q and K are arbitrary complex constants.

The substitution of (10) and (11) into (9) gives

$$-\frac{1}{4} = \frac{P^2}{4} + \frac{K^2}{\hbar^2}\delta_{0s} \tag{12}$$

and the following relations for determination of the parameters A and B of the confluent hypergeometric equation:

$$\frac{B}{2} - A = -\mathrm{i}P\frac{\mathrm{i}Q + s/2}{2} + \frac{K^2}{\hbar^2}\delta_{1s} \tag{13}$$

$$\frac{B}{2}\left(1 - \frac{B}{2}\right) = \frac{\mathrm{i}Q + s/2}{2}\left(1 - \frac{\mathrm{i}Q + s/2}{2}\right) + \frac{K^2}{\hbar^2}\delta_{2s} \tag{14}$$

where δ_{js} is the Kroneker symbol, $j, s = 0, 1, 2$.

Thus, we have found three classes of functions $U(t)$ and $\varepsilon(t)$, explicitly given by

$$U\,(t) = \frac{K}{z^{s/2}}\frac{\mathrm{d}z}{\mathrm{d}t} \qquad s = 0, 1, 2$$

$$\varepsilon(t) = \left(P + \frac{Q}{z}\right)\frac{\mathrm{d}z}{\mathrm{d}t} \tag{15}$$

for which the solution of the two-state problem can be expressed in terms of confluent hypergeometric functions. For chosen $z(t)$, the very solution is given by (4) and (5). The corresponding parameters A and B of the solution are given by equations (13) and (14). Note that equation (12) imposes an additional restriction on the parameters of (15). The explicit form of the factor φ (equation (5)) is as follows:

$$\varphi = z^{-(\mathrm{i}Q+s/2-B)/2}\mathrm{e}^{-(\mathrm{i}P+1)z/2}. \tag{16}$$

Let us now consider the transformation of the independent variable. Since equation (3) is invariant with respect to translation, $z \to z + z_0, z_0 =$ constant, it is clear from (12) (P is always an imaginary quantity) and (15) that the only non-trivial acceptable form of the variable transformation (i.e. leading to real U and ε) is

$$z = \mathrm{i}y(t). \tag{17}$$

Now one can find the pairs U–ε and the corresponding parameters of the confluent hypergeometric function, introducing different substitutions $y(t)$.

Consider some examples.

(i) $s = 0. \Rightarrow P^2 + 4(K/\hbar)^2 = -1$. If we choose

$$y = p\mathrm{e}^{-\gamma t} \qquad p\gamma = \sqrt{\alpha^2 + 4(U_0/\hbar)^2} \qquad K = \frac{\mathrm{i}U_0}{p\gamma} \qquad P = \frac{\mathrm{i}\alpha}{p\gamma} \qquad Q = -\frac{\Delta\epsilon}{\gamma}$$

then we obtain the class of Crothers and Hughes [6] which includes the function of Demkov [5],

$$U = U_0\,\mathrm{e}^{-\gamma t} \qquad \varepsilon = \alpha\,\mathrm{e}^{-\gamma t} + \Delta\epsilon.$$

(ii) $s = 1. \Rightarrow P = \mathrm{i}$. If we set $Q = 0$ and choose

$$y = a(t - t_0)^2 \qquad K = \frac{U_0}{2\sqrt{\mathrm{i}a}} \qquad a = -\alpha/2$$

then the system (15) presents the case of Landau–Zener [1, 2],

$$U = U_0 \qquad \varepsilon = \alpha(t - t_0).$$

(iii) $s = 2. \Rightarrow P = \mathrm{i}$. By the choice

$$y = \frac{\alpha}{\gamma}\mathrm{e}^{-\gamma t} \qquad K = -U_0/\gamma \qquad Q = -\Delta\epsilon/\gamma$$

one yields the class of Nikitin [4]:

$$U = U_0 \qquad \varepsilon = \alpha\,\mathrm{e}^{-\gamma t} + \Delta\epsilon.$$

Let us now return to the equation of invariants (9). It should be mentioned that the solution of the form (10) and (11) is not the only possible one. For instance, we can suggest the more general form, namely

$$\frac{U_z^*}{U^*} = \frac{s_1}{z} + \frac{s_2}{z+z_0} \qquad \varepsilon^* = P + \frac{Q}{z} + \frac{R}{z+z_0}.$$

Substituting these expressions into (9), one obtains $s_1 = -1/2$ and $s_2 = -1$, which lead to the following new class:

$$U = \frac{U_0}{\sqrt{y}(y+p)}\frac{\mathrm{d}y}{\mathrm{d}t} \qquad \varepsilon = \left(\varepsilon_0 + \frac{\varepsilon_1}{y} + \frac{\varepsilon_2}{y+p}\right)\frac{\mathrm{d}y}{\mathrm{d}t}$$
$$\varepsilon_0 p = \varepsilon_1 + \varepsilon_2/2 \qquad U_0^2/\hbar^2 = \frac{p}{4}(1+\varepsilon_2^2) \tag{18}$$

where $y(t)$ is an arbitrary real function, ε_0, ε_1, ε_2, p and U_0 are real constants.

The first independent solution of the initial equation (2) in explicit form is

$$a_1 = (y+p)^{-(\mathrm{i}\varepsilon_2+1)/2} \cdot {}_1F_1\left(-\frac{\mathrm{i}\varepsilon_2+1}{2}, \mathrm{i}\varepsilon_1 + \frac{1}{2}, -\mathrm{i}\varepsilon_0 y(t)\right)$$

where ${}_1F_1(A, B, z)$ is Pochhammer's function [9, 10]. The second independent solution is obvious.

In the case of constant ε, ε = constant = Δ, the obtained solution, (18), represents a new class of bell-shaped pulse functions (in magnetic resonance and optical resonance problems, this solution corresponds to the pulses of an external electromagnetic field having constant detuning, Δ, which vanishes at $t \to \pm\infty$):

$$U = \frac{U_0\sqrt{y}}{\gamma y^2 + (3\gamma p - \mu)y + \mu p}$$
$$\mathrm{e}^{\Delta t} = \mathrm{e}^{\gamma y} y^{\mu}(y+p)^{2(\gamma p - \mu)} \tag{19}$$
$$\varepsilon_0 = \gamma \qquad \varepsilon_1 = \mu \qquad \varepsilon_2 = 2(\gamma p - \mu) \qquad U_0^2/\hbar^2 = p(1/4 + (\gamma p - \mu)^2).$$

If we let $y = \exp(-\gamma t)$ and $p = 1$, then by the choice $\varepsilon_0 = -\alpha/\gamma$ and $\varepsilon_1 = -\Delta\epsilon/\gamma$ (18) becomes

$$U = \frac{\hbar\sqrt{\gamma^2/4 + (\alpha - \Delta\epsilon)^2}}{\mathrm{e}^{\gamma t/2} + \mathrm{e}^{-\gamma t/2}}$$
$$\varepsilon = \alpha\,\mathrm{e}^{-\gamma t} + \Delta\epsilon + \frac{2(\alpha - \Delta\epsilon)}{1 + \mathrm{e}^{\gamma t}}.$$

Here the modulation, ε, is analogous to that of the solution of Crothers and Hughes [6] (or Nikitin [4]) but the amplitude function, U, is that of Rosen–Zener [12]. Note that this solution includes the solution of Allen and Eberly [13] as a special case when $\alpha = 0$: $\varepsilon = \Delta\epsilon \cdot th(\gamma t/2)$, $U \sim \sec h(\gamma t/2)$.

In conclusion, we have suggested a new approach to finding analytically integrable cases of the two-state problem based on the equation of invariants (9) and the invariance property (8). The approach allows one to generalize all known analytically integrable cases, in terms of confluent hypergeometric functions, to a single formula (15). It also allows one to obtain a number of new classes of solution.

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